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ON MIXED CONVECTION MODES IN A VERTICAL LAYER WITH UNSTEADILY DEFORMABLE BOUNDARIES*

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The effect of three-dimensional periodic bending of boundaries on convection flows in a vertical layer is investigated in the case of mixed convection, the most important from the application point of view. Application of the proposed method enables to consider a more general formulation of the problem, when the channel walls perform oscillations. Since the bending amplitude of oscillations of the layer walls, the fluid flow velocity and the phase velocity of running waves at the boundaries are assumed small, the method of small perturbations is used. At Grashof numbers close to critical, four different modes of flow are revealed, and the stability regions and character of transition between them are determined. The obtained data indicate that the three-dimensional modulation of the boundary conditions may be used for controlling the stability of convection flows.

The convective heat exchange is effectively intensified by crimpling the heat transfer surfaces, by internal ribs in channels of heat exchanger /1/ and, also, by superposition of oscillations on the fluid flow /2/. Theoretical investigations of the effect of spatially inhomogeneous boundary conditions on the stability and the secondary modes of convective flows are few, and in the main concern free convection /3,4/.

A survey of the effect of active boundary deformation on isothermal flows appears in /5/.

1. Let an incompressible fluid perform a plane motion in an infinite vertical layer. The solid boundaries of the layer, the mean distance between which is $2d$, are maintained at temperatures $\mp\theta$ and are bent in opposite phase to sinusoidal law with amplitude ηd and period l. Variation of each boundary form with time is specified in the form of a running wave with the phase velocity C. The velocity stream along the layer averaged over the wave prior is maintained constant and equal Q. We describe the motion of fluid in dimensionless variables, selecting as the unit of length, stream function, time and temperature, respectively, $d, \nu, d^2/\nu$ and Θ. In the reference system moving along the vertical axis y with velocity C the equations of convection are of the form

$$\left(L-\frac{\partial}{\partial t}M\right)U+N(U,U)=0 \tag{1.1}$$

$$U=\begin{Vmatrix}\psi\\ T\end{Vmatrix},\quad L=\begin{Vmatrix}-\Delta^2 & G\dfrac{\partial}{\partial x}\\ 0 & P^{-1}\Delta\end{Vmatrix},\quad M=\begin{Vmatrix}-\Delta & 0\\ 0 & 1\end{Vmatrix}$$

$$N(U,U)=\begin{Vmatrix}-\partial(\psi,\Delta\psi)/\partial(x,y)\\ \partial(\psi,T)/\partial(x,y)\end{Vmatrix},\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

$$G=\frac{g\beta\Theta d^3}{\nu^2},\quad P=\frac{\nu}{\chi},\quad c=\frac{Cd}{\nu}$$

where ψ is the stream function, T is the temperature, G is the Grashof number, P is the Prandtl number, and c is the dimensionless phase velocity of the wave at the boundary.

Assuming that the displacement of walls is purely transverse and the flow three-dimensionally periodic, we write the boundary conditions (in the moving reference system) as /6/

$$x=-(1+\eta\cos ky),\ T=1,\ \psi=q-2c,\ \partial\psi/\partial x=c$$
$$x=1+\eta\cos ky,\ T=-1,\ \psi=0,\ \partial\psi/\partial x=c$$
$$U(x,y+2\pi/k)=U(x,y);\ k=2\pi d/l,\ q=Q/\nu$$

We introduce the coordinate transformation which would rectify the curved boundaries of the layer /3,6/

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$$y' = y,\ x' = x/(1 + \eta \cos ky)$$

The explicit form of equations of convection in new variables is given in /3/. Assuming parameter η to be small, we represent the equations in the form of series in η /4/

$$\left(L' - \frac{\partial}{\partial t} M'\right) U + N'(U, U) = \sum_{n=1}^{\infty} \eta^n \sum_{m=-n}^{n} \left[\left(L_{mn} - \frac{\partial}{\partial t} M_{mn}\right) U + N_{mn}(U, U)\right] e^{imky'} \quad (1.2)$$

$$L_{-m,n} = \bar{L}_{mn}, \quad M_{-m,n} = \bar{M}_{mn}, \quad N_{-m,n} = \bar{N}_{mn}$$

where L', M', N' are the operators L, M, N in which the substitution $\partial/\partial y \to \partial/\partial y'$, $\partial/\partial x \to \partial/\partial x'$ has been carried out, and L_{mn}, M_{mn}, N_{mn} are operators whose explicit form is not presented because of unwieldiness. The boundary conditions in new variables are stated on plane boundaries.

$$x' = -1,\ T = 1,\ \psi = q - 2c,\ \partial\psi/\partial x' = c\,(1 + \eta \cos ky') \quad (1.3)$$
$$x' = 1,\ T = -1,\ \psi = 0,\ \partial\psi/\partial x' = c\,(1 + \eta \cos ky')$$

The primes are henceforth omitted for brevity.

2. When $\eta = q = c = 0$ the problem (1.2), (1.3) has always the solution

$$U^{(0)} = (\psi_0, T_0), \quad \psi_0 = -\frac{G}{24}(1 - x^2)^2, \quad T_0 = -x \quad (2.1)$$

which corresponds to a plane-parallel motion. The threshold Grashof number $G_0(k)$ when $P < 12$ is increased, this solution becomes unstable with respect of monotonically increasing perturbations of period $2\pi/k$ which results in the appearance of steady solutions periodic with respect to y in conformity with the general theory of secondary motion generation /7,8/.

Below, we investigate the properties of solution branching when G is close to $G_0(k)$ in the case of small, but nonzero η, q and c. As in /4/ we seek a solution of problem (1.2), (1.3) in the form of series in the small parameter ε that defines the order of smallness of the nonplane-parallel component of motion

$$U - U^{(0)} = \sum_{n=1}^{\infty} \varepsilon^n U^{(n)} \quad (2.2)$$

Setting $G - G_0 = O(\varepsilon^2)$, we introduce the notation

$$G - G_0 = \varepsilon^2 G^{(2)} \quad (2.3)$$

We represent the relation between quantities ε and η in the form

$$\eta = \sum_{n=1}^{\infty} \varepsilon^n \eta^{(n)} \quad (2.4)$$

where the coefficients $\eta^{(n)}$ are to be determined from the condition of solvability of the boundary value problem of the n-th order with respect to ε. For the analysis of time evolution of motions we apply the method of many scales /9/. Functions $U^{(n)}$ in expansion (2.2) are assumed dependent on the set of variables $t_n = \varepsilon^n t$, and we carry out in Eq.(1.2) the substitution

$$\frac{\partial}{\partial t} = \sum_{n=1}^{\infty} \varepsilon^n \frac{\partial}{\partial t_n} \quad (2.5)$$

The quantities c and q are independent parameters. (In selecting as independent parameters η, c and q, the pressure drop on the wave length is a function of these parameters /6/). Nevertheless we shall consider in the beginning the case when c and q are, together with $G - G_0$, of order ε^2

$$c = \varepsilon^2 c^{(2)}, \quad q = \varepsilon^2 q^{(2)} \quad (2.6)$$

In this case the time scales which determine the perturbation amplitude growth and the change of phase (which are defined by the increment and frequency of perturbations), are of the same order. At the same time the equation that defines the secondary motions proves to be the most interesting. The nonfulfillment of relations (2.6) is discussed in Sect.4.

Let us substitute (2.2)–(2.6) into (1.2) and (1.3) and require the solvability of the boundary value problem in the class of functions that are bounded as $y \to \pm\infty$, $t \to \infty$ in every order with respect to ε. In the first order we have $\eta^{(1)} = 0$, and the solution is of the form

$$U^{(1)} = a_1 u_1^{(1)}(x) e^{iky} + \bar{a}_1 \bar{u}_1^{(1)}(x) e^{-iky} \qquad (2.7)$$

where $u_1^{(1)} = (\varphi_1^{(1)}, \theta_1^{(1)})$ is a function that describes the neutral perturbation of the plane-parallel flow in the problem with plane boundaries $a_1 = a_1(t_1, t_2, \ldots)$. In the second order with respect to ε we obtain

$$\eta^{(2)} = 0, \quad \partial a_1/\partial t_1 = 0, \quad U^{(2)} = 2\,\mathrm{Re}\,[u_2^{(2)}(x) e^{2iky} a_1^2] + u_0^{(2)}(x)|a_1|^2 + c^{(2)} u_c(x) + q^{(2)} u_q(x)$$

$$u_c(x) = \begin{Vmatrix} x-1 \\ 0 \end{Vmatrix}, \quad u_q(x) = \begin{Vmatrix} \frac{1}{4}(x^3 - 3x + 2) \\ 0 \end{Vmatrix}$$

where functions $u_2^{(2)}(x)$ and $u_0^{(2)}(x)$ are defined in /10/. Finally in the third order with respect to ε we obtain the equation of amplitude a_1. Multiplying it by ε^3 we obtain the approximate equation for the quantity $a = \varepsilon a_1$ that determines the nonlinear evolution of perturbation of form (2.7)

$$\frac{da}{dt} = (\sigma_r + i\sigma_i) a - \varkappa |a|^2 a + d\eta \qquad (2.8)$$

The equation of type (2.8) was considered in the investigation of resonance excitation and oscillation of Van der Pol oscillator by an external periodic force /9/. The coefficients $\sigma_r, \varkappa, d$ are connected to quantities I, J, S and D of /4/ by the relations

$$\varkappa = S/I, \quad d = D/I, \quad \sigma_r = J(G - G_0)/I$$

according to data in /4/ when $P = 1$, $k = 1.38$ we have

$$\sigma_r = 2.3 \cdot 10^{-2}(G - G_0), \quad \varkappa = 1.1 \cdot 10^3, \quad d = 7.3$$

where the normalization of function $\mathrm{Re}\,\theta_1^{(1)\prime}(-1) = 1$ is selected. The quantity $\sigma_i = ck + bq$ is recalculated, with the transition to the moving reference system taken into account, the imaginary part of the increment of the plane parallel perturbation of mixed convection flow with a given rate of flow, whose stability was earlier investigated in /11,12/. According to these data, the quantity b for the same values of parameters is -0.6.

We set

$$a = z(|\sigma_i|/\varkappa)^{1/2}, \quad \tau = t|\sigma_i|, \quad \sigma_r = \gamma|\sigma_i|, \quad |d|\eta = \delta(|\sigma_i|^3/\varkappa)^{1/2} \qquad (2.9)$$

In new variables Eq.(2.8) assumes the form

$$z^{\cdot} = (\gamma + i\,\mathrm{sign}\,\sigma_i) z - |z|^2 z + \delta\,\mathrm{sign}\,d \qquad (2.10)$$

where the dot denotes differentiation with respect to τ. In what follows we assume $\sigma_i > 0$, $d > 0$ for definiteness. All results are transferred in an obvious way to the case of opposite signs of σ_i and d. All quantities appearing in Eq.(2.10) are of order unity.

Equation (2.10) for the complex variable z is equivalent to the system of two real equations. The limit modes for such system can be stationary points, cycles, and separatrix contours /13/.

We shall use two representations of Eq.(2.10) in the form of a system of real equations. Setting $X = \mathrm{Re}\,z$, $Y = \mathrm{Im}\,z$ we write (2.10) in the form

$$X^{\cdot} = (\gamma - X^2 - Y^2) X - Y + \delta, \quad Y^{\cdot} = (\gamma - X^2 - Y^2) Y + X \qquad (2.11)$$

Note the system (2.11) also defines the phenomena near the threshold of convection onset in porous medium (*).

Setting $r = |z|$, $\varphi = \arg z$ it is possible to reduce (2.10) to the form

$$r^{\cdot} = \gamma r - r^3 + \delta\cos\varphi, \quad r\varphi^{\cdot} = r - \delta\sin\varphi \qquad (2.12)$$

where the meaning of r is that of fundamental mode inducing instability in the case of plane boundaries and φ are phases that determine the position of vortex centers relative to the boundaries ($\varphi = 0$ corresponds the vortex center location in the widest, and $\varphi = \pi$ in the narrowest cross section). We stress that in the case of $r = 0$ the flow is not plane-parallel, but in expansion (2.2) the term (2.7), which defines the input of the basic mode, is absent.

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*) Liubimov D.V., Certain problems of convective stability in porous medium. Candidate Fiz.-Matem. Nauk Dissertation. Perm'skii Gos. Universitet, 1979.

3. Let us pass to the investigation of limit modes of system (2.10) and their stability. First, we shall consider the time independent solutions of system the motions that are stationary in the moving reference system (convection vortices are carried along by the peristaltic motions of the walls). From (2.12) we obtain the relations

$$\gamma = r^2 \pm \sqrt{\delta^2/r^2 - 1}, \quad \operatorname{tg}\varphi = -(\gamma - r^2)^{-1} \tag{3.1}$$

The dependence of r^2 and φ on γ is shown in Figs.1 and 2 for $\delta = 1$, $\delta = 1.3$ and $\delta = 1.5$ lines 1,2, and 3). When $\delta < \delta_1 = 2^{3/2}3^{-3/4} \approx 1.24$, then for any value of γ we have a unique stationary solution of system (2.12); with the increase of γ the quantity r^2 first increases, reaching the values $r^2 = \delta^2$ at $\gamma = \delta^2$, and then decreases; the phase φ varies between 0 as $\gamma \to -\infty$ and π as $\gamma \to +\infty$. If in some interval of values of γ, $\gamma_- < \gamma < \gamma_+$ $\delta > \delta_1$, the system has three steady solutions (corresponding to sections I, II, and III of curves 2 and 3 in Figs.1 and 2). It can be shown that $\gamma_\pm$ are linked with δ by the relation

$$\delta^2 = \frac{2}{27}[\gamma_\pm(\gamma_\pm^2 + 9) \mp (\gamma_\pm^2 - 3)^{3/2}]$$

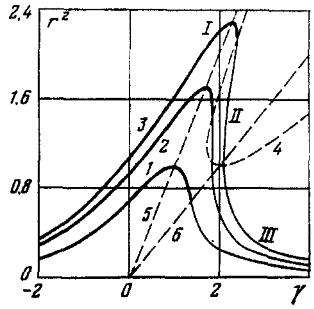

Fig.1

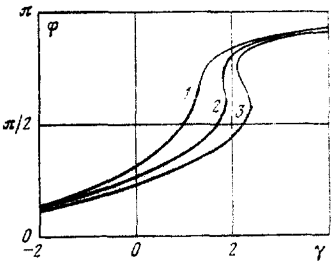

Fig.2

(lines *1* and *2* in Fig.3). The branch merging points $d\gamma/dr^2 = 0$ lie on the line

$$\gamma = 2r^2 \pm \sqrt{r^4 - 1}$$

(line *4* in Fig.1); the maximum values of r^2 for a given δ is reached for $\gamma = r^2$ (line *5* in Fig.1), and then $\varphi = \pi/2$.

Let us investigate the stability of steady solutions. Linearizing system (2.12) near solution (3.1), we obtain the expression for perturbation increments

$$\lambda_\pm = \gamma - 2r^2 \pm \sqrt{r^4 - 1} \tag{3.2}$$

or taking into account (3.1)

$$\lambda_\pm = \gamma - 2r^2 \pm \left[(\gamma - 2r^2)^2 + 2r^2(\gamma - r^2)\frac{d\gamma}{dr^2}\right]^{1/2} \tag{3.3}$$

The last formula shows that steady motion is always unstable (since $\lambda_+ > 0$), when $(\gamma - r^2)\, d\gamma/dr^2 > 0$, which takes place for section II of curve $r^2(\gamma)$. Thus section II is always unstable (saddle points correspond to it), and the instability is of a monotonic character. It follows from formula (3.2) that for $\gamma = 2r^2$ (line *6* in Fig.1) solution (3.1) loses its stability in an oscillatory manner, if with this $r^4 < 1$. This is realized at $\delta < \delta_2 = 2^{1/2} \approx 1.41$, with the boundary of oscillatory instability γ_0 (line *3* in Fig.3) is defined by the formula

$$\delta^2 = \gamma_0(\gamma_0^2 + 4)/8$$

The stable sections of curves $r^2(\gamma)$, $\varphi(\gamma)$ are shown in heavy lines in Figs.1 and 2. With $\delta < \delta_1$ and increasing γ the solution loses stability in an oscillatory manner at $\gamma = \gamma_0$ (line *3* in Fig.3). When $\delta > \delta_1$ the branch I is stable throughout the region of its existence $\gamma < \gamma_+$ (to the left of line *2* in Fig.3). In the region $\delta_1 < \delta < \delta_2$ branch III has a section of stability (between lines *1* and *3* in Fig.3). Lines *2* and *3* intersect at point P_3 with coordinates $\gamma_3 \approx 1.79$, $\delta_3 \approx 1.27$.

In addition to stationary points the limit mode for system (2.10) can be a cycle, which is shown in Fig.3 by line *3* that is the branching boundary of the cycle with low amplitude and frequency, determined by formula (3.2). Calculation of the branching constant shows that along the whole oscillatory boundary of instability the cycle branching occurs softly, (in the direction of large γ), and the cycle is stable. At point P_2 in Fig.3 ($\gamma_2 = 2$, $\delta = \sqrt{2}$) at which the monotonic and oscillating instability merge, bifurcation of codimension 2 (two zero roots) that was described in /14/.

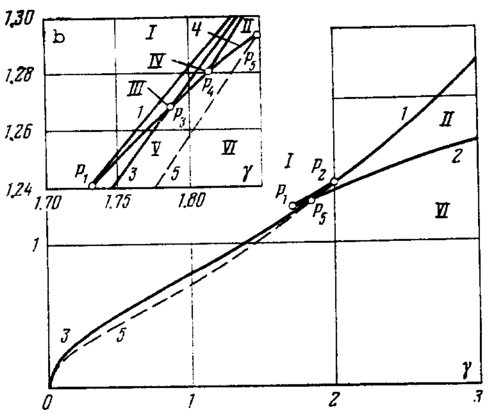

Fig.3

Apart the described above bifurcation boundaries 1 and 3, at point P_2 terminates the bifurcation line *4* on which there exists the separatrix ("homoclinal trajectory") doubly asymptotic to the saddle point that coresponds to steady solution on branch II. The boundary 4 was constructed using numerical integration of system (2.12) by the Runge-Kutta method. When approaching boundary 4 from the side of smaller γ, the cycle period approaches infinity in conformity with the logarithmic law. In region II the cycle is absent between lines *2* and *4*. Line 4 terminates at point P_4 ($\gamma_4 \approx 1.82, \delta_4 \approx 1.28$) on line *2*, on which the saddle point becomes a saddle-node. When $\gamma > \gamma_4$ on line *2* we have a loop of the saddle-node separatrix, from which, as γ increases, a cycle of finite amplitude is generated; its period on approaching boundary 2 tends to infinity in proportion to $(\gamma - \gamma_+)^{-1/2}$. If $\gamma < \gamma_4$, new limit modes are not generated at the disappearance of the saddle-node(*).

A cycle, thus, corresponds to limit mode of the system as $t \to \infty$ in the region bounded by lines *3*, *4*, and *2* in Fig.3, when in region IV are simultaneously stable the cycle and the steady solution (on branch I). On line *3* the amplitude of cycle vanishes (for finite period), while on lines *2* and *4* the cycle period becomes infinite (for finite amplitude).

Let us consider the pattern of motion of the respective cycle. As long as the cycle amplitude is not large (region IV and V in Fig.3), the order of trajectories on plane (X, Y) corresponding to the cycle, relative to the point $r = 0$ is zero, and the variable φ is a periodic function of time. Such solution defines the periodic oscillations of vortex center relative to some mean position. On some line *5* (Fig.3) the cycle passes through point $r = 0$. The boundary 5 was determined by numerical integration of system (2.11); it touches line *3* at point $\gamma = \delta = 0$ and terminates on line *2* at point P_5 with coordinates $\gamma_5 \approx 1.86, \delta_5 \approx 1.30$.

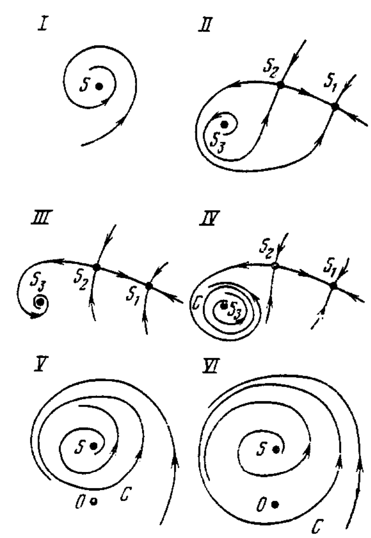

Fig.4

In region VI the order of cycle relative to point $r = 0$ becomes equal unity. The phase φ increases to 2π over the cycle period T_c, and the solution corresponds to a wave running relative to the walls with phase velocity $2\pi/kT_c$. Thus two types of motion correspond to the cycles: oscillations of the vortex and the running waves. Note that the transition between these two types of motion is not a bifurcation, and is related only to the change of the cycle position relative to point $r = 0$ which is not singular.

Thus it is possible to separate in the parameter plane (γ, δ) six regions (Fig.3). In region I exists and is stable a unique steady motion. In region II there are three stationary points of which only one corresponds to a stable motion. In region III two out of three of the singular points, corresponding to different amplitudes and positions of the vortex relative to the walls, are stable. In region IV are stable one of the steady solutions and the cycle that corresponds to oscillations of the vortex with respect to phase and amplitude. In regions V and VI the single steady solution of the system is unstable, while the cycle is stable, and in region V it corresponds to vortex oscillations, and in region VI to a continuous running wave. The phase patterns for all these regions are schematically shown in Fig.4. Stationary points related to branches I, II, and III are denoted by S_1, S_2, S_3, the

*) Aponin Iu.M., Asymptotic formulas for the limit cycle at degeneration into a loop of the saddle-node separatrix. Preprint, n.-i. VTs, Akad. Nauk SSSR, Pushchino, 1980.

single stationary point by S, and $r = 0 - O$, the cycle point by C.

4. Let us discuss the character of change of flow mode in the case, when the relations (2.6), that establishes and the real and imaginary parts of increment σ_r and σ_i are of one order, is not satisfied.

If $c \gg \varepsilon^2$ or $q \gg \varepsilon^2$, then $\sigma_i \gg \sigma_r$; according to (2.9), this case is suitable to the limit $\gamma \to 0$, $\delta \to 0$. The boundary of oscillating instability of steady solution in this limit assumes the form $\gamma = 2\delta^2$, or

$$\sigma_r = 2\varkappa d^2\eta^2/\sigma_i^2$$

To determine the asymptotics of boundary 5 we represent the solution of system (2.11) in the form of series in powers of the small parameter δ

$$X = \sum_{n=1}^{\infty} \delta^n X_n, \quad Y = \sum_{n=1}^{\infty} \delta^n Y_n$$

setting $\gamma = \Gamma\delta^2$. Functions X_n and Y_n are assumed dependent on time $\tau_m = \delta^m\tau$, $m = 0, 1, \ldots$.

In the first order with respect to δ we have

$$X_1 = A \cos(\tau_0 + \varphi), \quad Y_1 = 1 + A \sin(\tau_0 + \varphi) \tag{4.1}$$

where A and φ may be functions of slow time. The condition of solution boundedness we have in second order $\partial A/\partial\tau_1 = \partial\varphi/\partial\tau_1 = 0$, while in the third order we have

$$\partial A/\partial\tau_2 = A(\Gamma - 2 - A^2), \quad \partial\varphi/\partial\tau_2 = 0$$

From this follows that the equation of cycle is of the form (4.1), where $A = \sqrt{\Gamma - 2}$. The cycle passes through the point $X = Y = 0$ when $\Gamma = 3$ so that the asymptotics of boundary 5 at γ as $\delta \to 0$ has the form $\gamma = 3\delta^2$, which yields the boundary of transition from oscillations to running waves

$$\sigma_r = 3\varkappa d^2\eta^2/\sigma_i^2$$

This result is confirmed by direct numerical integration of system (2.11).

If c and q are considerably smaller than ε^2, it is necessary to omit in Eq. (2.8) the term with σ_i, which brings us to the case considered earlier in /4/. The stable motion is then steady.

5. In conclusion we shall consider the basic physical consequences of obtained results in the case of pumping fluid through a channel with stationary walls (the case of nonstationary boundaries may be similarly treated). Mixed convection flow with a given discharge rate is steady, as long as the Grashof number does not exceed a certain threshold value. With further increase of temperature difference the flow spontaneously becomes unsteady. The violation threshold of flow steadiness is heightened, as the bending amplitude of the boundary increases and the flow rate of fluid decreases; the smooth bending of the boundary, thus, not only increases the heat exchange but, also, stabilizes the flow. The nature of unsteadiness close to the threshold materially depends on the value of bending parameter δ. When $\delta < \delta_1$, small oscillations of vortices that are realized, continuously pass with increasing Grashof number, to straight-through motion of vortices. When $\delta > \delta_4$ the unsteady motion arises in the form of oscillations ($\delta_4 < \delta < \delta_5$), or through motion of vortices ($\delta > \delta_5$) with a very long time period defined by a sequence of prolonged time intervals during which the vortex position slowly changes (the phase trajectory passes close to a saddle point), alternating with short intervals of rapid rearrangement of flow. In region $\delta_1 < \delta < \delta_4$ are possible hysteresis effects between various flow modes. The diversity of flow modes and the existence of hysteresis effects open possibilities of an effective control of flow modes.

In the case of unsteady boundaries a system of vortices completely carried away by the peristaltic motion of the boundary, correspond to steady solution in the stationary reference system. To a solution in the form of oscillations corresponds a system of vortices moving at phase velocity of the wall and simultaneously oscillating; to a solution of a running wave type corresponds a system of vortices moving at mean phase velocity, different from the wall phase velocity.

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